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Hierarchical Models and Chaotic Spin Glasses

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Renormalization-group studies in position space have led to the discovery of hierarchical models which are exactly solvable, exhibiting nonclassical critical behavior at finite temperature. Position-space renormalization-group approximations that had been widely and successfully used are in fact alternatively applicable as exact solutions of hierarchical models, this realizability guaranteeing important physical requirements. For example, a hierarchized version of the Sierpiński gasket is presented, corresponding to a renormalization-group approximation which has quantitatively yielded the multicritical phase diagrams of submonolayers on graphite. Hierarchical models are now being studied directly as a testing ground for new concepts. For example, with the introduction of frustration, chaotic renormalization-group trajectories were obtained for the first time. Thus, strong and weak correlations are randomly intermingled at successive length scales, and a new microscopic picture and mechanism for a spin glass emerges. An upper critical dimension occurs via a boundary crisis mechanism in cluster-hierarchical variants developed to have well-behaved susceptibilities.

KEY WORDS: Hierarchical models; renormalization group; Sierpiński gasket; frustration; chaos; spin glass; boundary crisis.

A hierarchical model[1,2] is constructed by replacing a bond, representing the interaction between degrees of freedom, by a graph (or graphs[3]) of such bonds, and repeating this process *ad infinitum* (Fig. 1a). The outcome of this repeated self-embedding is a thermodynamic system which is exactly solvable. The solution is a renormalization-group procedure which, at least graphically, inverts the self-embedding construction. The interaction strength of the bonds of course in general changes during each renormalization-group step, and new types of interaction associated with the bond may be generated, depending upon the symmetry of the degrees of freedom. These models, solved exactly, can exhibit finite-temperature phase

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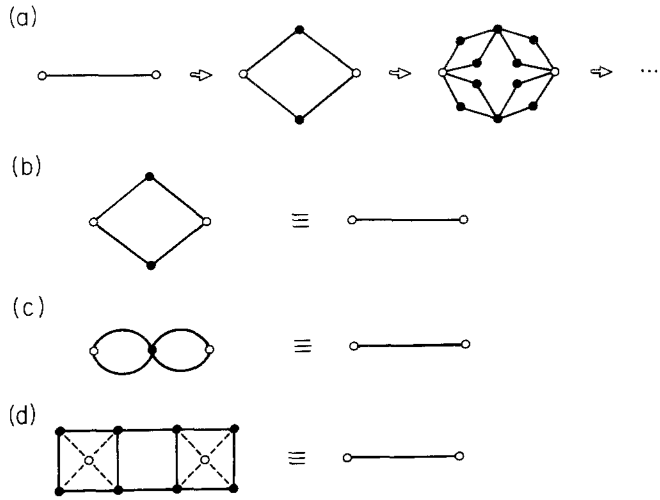

Fig. 1. Construction (a) of the hierarchical model (Ref. 1) corresponding to a Migdal–Kadanoff recursion (decimation followed by bond-moving), and its representation (b) by a graphical identity. (c) The hierarchical model corresponding to the other Migdal–Kadanoff recursion (bond-moving followed by decimation). (d) A cluster-hierarchical model (Ref. 19).

transitions that are nonclassical, as can be understood by the fact that they contain closed loops.

The evolution, under renormalization group, of the interaction strengths is embodied in the recursion relations.[4] When the additive constant to the Hamiltonian is included,[5] these relations contain the entire statistical mechanics of the system under consideration. In some cases, the recursion relations for the approximate treatment of an ordinary lattice are simultaneously applicable as the exact solution of an appropriate hierarchical lattice. This correspondence is in fact how these hierarchical models were introduced[1] and has a general implication for the approximate treatments that possess such correspondence. Namely, these approximations are physically realizable, be it on rather unusual alternate systems (see below), and can be *a priori* identified to give thermodynamically sensible results.[1] For example, the specific heat will always be positive. It was also noted, in hindsight, that most position-space renormalization-group approximations that had been widely and successfully used do indeed have hierarchical realizations. To give a few examples, Fig. 1a corresponds to one of the Migdal–Kadanoff[6] recursion relations. Its self-embedding can also be represented by the graphical identity of Fig. 1b. Figure 1c represents the other Migdal–Kadanoff recursion relation. Figure 1d represents a hierarchical model akin (but not identical, owing to a different volume rescaling factor,[7] as defined below) to the Niemeijer–van Leeuwen cluster approximation.[8] Figure 2b represents the hierarchical model corresponding to the Kadanoff bond-moving approximation.[9]

In Fig. 2a, the shaded triangles contain interactions between degrees of freedom at the small circles. Three right-pointing triangles form a 'unit"; $n = 2$ units are joined in parallel and form the self-embedding graph. With $n = 4/3$, this renormalization-group transformation also applies to the plaquette-moving aproximation[10] for krypton and nitrogen submonolayers on basal graphite. Using Potts-lattice-gas degrees of freedom, demanding 22 different types of interaction within the shaded triangles coupling spin and space directions, this treatment yielded[10] multicritical phase diagrams within a few percent of experimental observations. For $n = 1$ this model reduces to the Sierpiński gasket, which was previously discussed[11] in a geometrical context and subsequently studied[12] as a phase transition problem. In that special case, the model looks less unusual, since the $n > 1$ padding has been removed. On the other hand, only a zero-temperature phase transition is obtained owing to the finite ramification.[12]

An "effective" dimensionality d can be defined[13] for hierarchical models: The length rescaling factor b is the number of bonds in the shortest path between the external vertices of the self-embedding graph (open circles in the figures). The volume rescaling factor b^d is the total number of bonds in the self-embedding graph. Note that d plays no direct role in the exact solution of the hierarchical model. It does serve a bookkeeping role in a family of models and incorporates qualitative trends associated with ordinary dimensionality. Also, note that the spatial locations of the degrees of freedom have not been specified in hierarchical models, and need not be. On the other hand, a hierarchical model may be visualized on an ordinary lattice, provided one allows some zero and infinite interactions.[14]

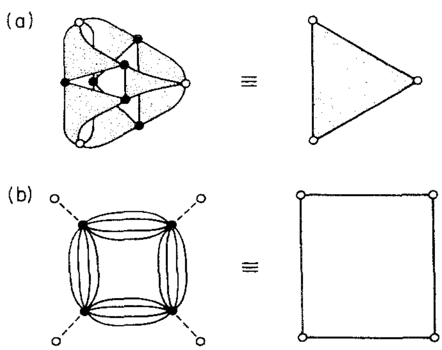

Fig. 2. The hierarchized Sierpiński gasket, with $n = 2$ units in parallel. In Ref. 10, $n = 4/3$ was used. (b) The hierarchical model corresponding to the Kadanoff bond-moving approximation. Dashed lines represent projection operators.

Hierarchical models are now being studied directly as a testing ground for new concepts,[13,15,16] without invoking correspondence to an approximate position-space renormalization group. One such study[13,3,15] involves the introduction of frustration.[17] Before describing some results, one more aspect of hierarchical models need be noted. Although the average coordination number is finite for a model such as in Fig. 1a, it is clear that sites at different levels of the hierarchy have different coordination numbers. Specifically, a smaller and smaller number of sites have a larger and larger coordination number. The highly coordinated sites dominate the long length scales. When the site–site interaction is nonzero, these sites are very susceptible to an external field, whose effect they feel channeled through their many neighbors. Thus, an infinite susceptibility is seen at all temperatures in the paramagnetic phase,[18] except literally at infinite temperature. This physically unreasonable characteristic does not occur in so-called cluster-hierarchical models involving a projection operator (e.g., Fig. 1d), which avoid the high coordinations and exhibit well-behaved susceptibilities along the entire temperature range.[19]

Accordingly, consider the frustrated cluster-hierarchical model[15] defined by Fig. 3. Each site has an Ising degree of freedom $s_i = \pm 1$. The full straight lines represent ferromagnetic interactions $-\beta\mathscr{H}_{ij} = Ks_i s_j$. The wiggly lines, representing infinite antiferromagnetic interactions, cause frustration.[13] The dashed lines represent projection operators.[19] In the reentrant unit (Fig. 3a), at low temperatures, correlations across the unit are completely destroyed by the short-range correlations across the p cross-bonds. In the repressed unit (Fig. 3b), correlations across the shorter path are not completely destroyed but are substantially reduced by the competing

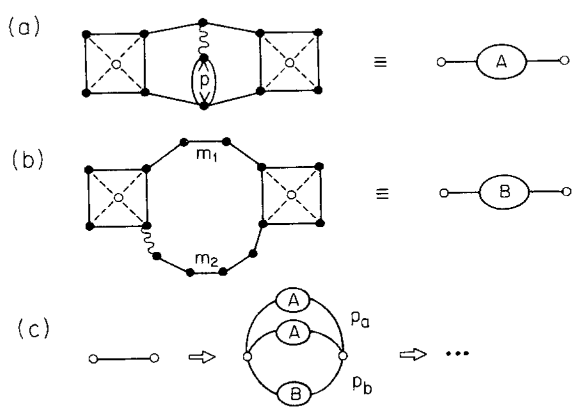

Fig. 3. The frustrated cluster-hierarchical model (Ref. 15).

longer path. These two units are taken to represent[13] the generically different microscopic geometries in a real spin glass. Respectively, p_a and p_b of these units are joined in parallel to obtain the self-embedding graph (Fig. 3c). The built-in tendency to disorder can be increased by increasing either the participation of the reentrant unit (p_a), or the ground state degeneracy of the repressed unit (m_1). In either case, the effective dimensionality

$$d = \ln[p_a(12 + p) + p_b(8 + m_1 + m_2)]/\ln 3$$

is increased.[15]

A typical set of results is depicted in Fig. 4. At an increased built-in tendency to disorder, the model exhibits chaotic renormalization-group trajectories. This has been,[13] to our knowledge, the first time that such trajectories are obtained, and has direct physical interpretation: As the system is probed at successive length scales, strong and weak correlations are encountered in a chaotic sequence. This implies an ordered phase with infinite interpenetrating subsets of noncontiguous yet strongly correlated spins.[13] A most recent result[15] in these studies has occurred with the introduction of cluster-hierarchical[19] models. As dimensionality is

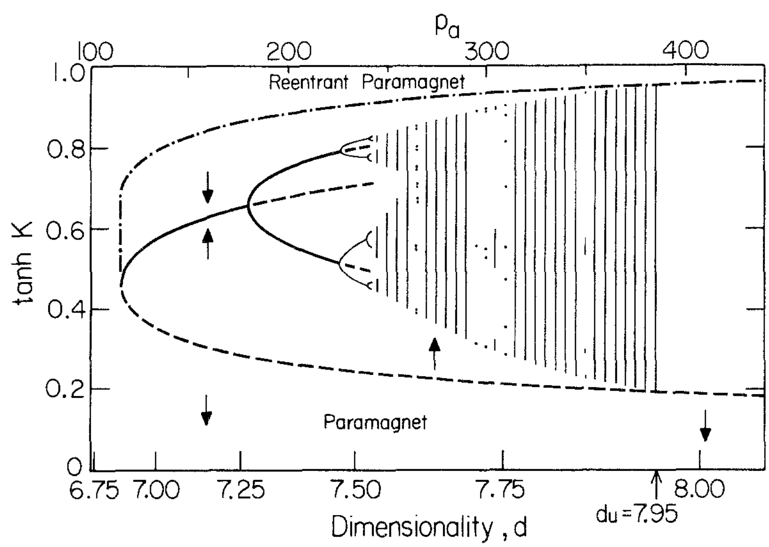

Fig. 4. Renormalization-group flows for $(p_b, p, m_1, m_2) = (2, 4, 6, 7)$ in the frustrated cluster-hierarchical model. The stable (unstable) fixed point and some of the stable (unstable) limit cycles are shown with full (dashed) lines. The dash-dotted line is the image of the lowest dashed line. Some of the chaotic bands are shown by the vertical segments. Arrows depict flows.

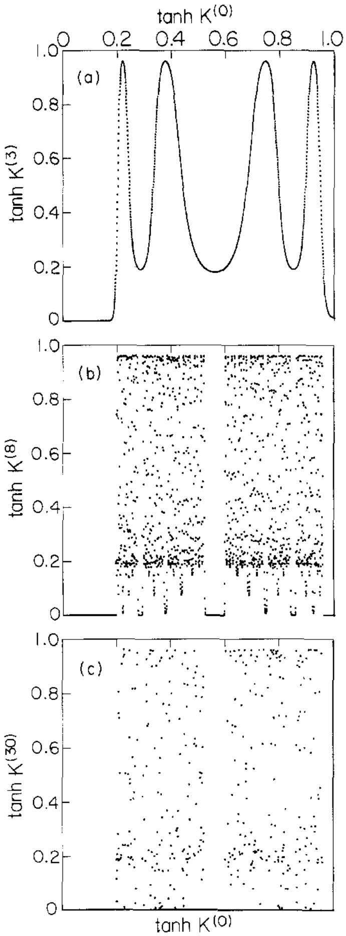

Fig. 5. Renormalization-group iterations of 2000 initial interactions for the system of Fig. 4 with $d = 8$, that is, under intermediate-range chaotic spin-glass conditions. Even after 30 iterations, many trajectories have not escaped the chaotic band, which is clearly accentuated after eight iterations. The bottom horizontal axes have not been traced, to reveal those trajectories that have essentially reached the decoupled fixed point.

increased, the chaotic band expands and eventually annihilates via a boundary crisis, when its edge collides with the unstable critical fixed point. Beyond this upper critical dimension, the entire temperature range eventually renormalizes to the decoupled fixed point, and therefore is paramagnetic. However, the continuation of the chaotic spin-glass phase exhibits considerable intermediate-range order, as seen in Figs. 5. The renormalization-group iterations of 2000 initial interactions are shown. After eight iterations, which is a huge length rescaling ($\times 3^8$), many of these trajectories have not escaped the chaotic band. Some remain even after 30 iterations.

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